I5-0910: 2. Primjena Stokesovog teorema

Provjerite ispravnost Stokesovog teorem o rotaciji za dano vektorsko polje

$$\vec{F} = 3y\hat{\imath} - xz\hat{\jmath} + yz^2\hat{k}$$

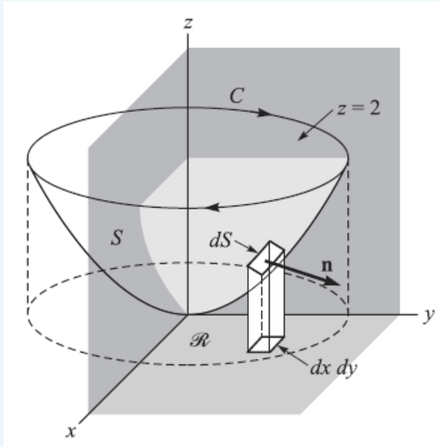
i plohu S, koja je donji dio paraboloida

$$2z = x^2 + y^2$$

čiju graničnu krivulju C dobijemo ako ga presiječemo plohom

$$z = 2$$

kao na slici.



Rub isječenog paraboloida je krivulja

$$x^2 + y^2 = 4$$
$$z = 2$$

koju možemo parametrizirati na sljedeći način $(0 \leq \varphi < 2\pi)$

$$x = 2\cos\varphi$$
$$y = 2\sin\varphi$$
$$z = 2$$

Izračunajmo integral po gore spomenutoj zatvorenoj krivulji

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \left(3y\hat{\imath} - xz\hat{\jmath} + yz^2\hat{k}\right) \cdot \left(dx\hat{\imath} + \mathrm{d}y\hat{\jmath} + \mathrm{d}z\hat{k}\right) = \oint_C (3y\mathrm{dx} - xz\mathrm{dy} + yz^2\mathrm{dz}) =$$

$$= \oint_{2\pi}^{0} [3(2\sin\varphi)(-2\sin\varphi)\mathrm{dt} - 2\cos\varphi \cdot 2 \cdot (2\cos\varphi)\mathrm{d\varphi} + (2\sin\varphi)(2^2) \cdot 0] =$$

$$= \oint_{0}^{2\pi} [12\sin^2\varphi + 8\cos^2\varphi]dt = \oint_{0}^{2\pi} [8\sin^2\varphi + 8\cos^2\varphi]d\varphi + 4\oint_{0}^{2\pi} \sin^2\varphi \, d\varphi =$$

$$= 8\oint_{0}^{2\pi} d\varphi + 4\oint_{0}^{2\pi} \sin^2\varphi \, d\varphi = 8\oint_{0}^{2\pi} d\varphi + 4\oint_{0}^{2\pi} \frac{1 - \cos 2\varphi}{2} d\varphi =$$

$$= 16\pi + 4\oint_{0}^{2\pi} \frac{1}{2} d\varphi - 4\oint_{0}^{2\pi} \frac{\cos(2\varphi)}{2} \cdot \frac{1}{2} d(2\varphi) =$$

$$= 16\pi + 4\pi - 4 \cdot 0 = 20\pi$$

Kako je krivulja C rub isječenog paraboloida, možemo primijeniti Stokesov teorem za čije nam je računanje potrebno:

1. Normala (dani paraboloid možemo promatrati kao nivo plohu skalaranog polja $\Phi(x, y, z) = x^2 + y^2 - 2z$, a gradijent je uvijek okomit na nivo plohe pripadnog skalarnog polja)

$$\hat{n} = \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{x\hat{\imath} + \mathrm{y}\hat{\jmath} - \hat{k}}{\sqrt{x^2 + y^2 + 1}}$$

2. Rotacija polja $\vec{F}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial \mathrm{x}} & \frac{\partial}{\partial \mathrm{y}} & \frac{\partial}{\partial \mathrm{z}} \\ 3\mathrm{y} & -\mathrm{xz} & yz^2 \end{vmatrix} = \hat{\imath}\left(\frac{\partial(yz^2)}{\partial \mathrm{y}} - \frac{\partial(-\mathrm{xz})}{\partial \mathrm{z}}\right) + \hat{\jmath}\left(\frac{\partial(3\mathrm{y})}{\partial \mathrm{z}} - \frac{\partial(yz^2)}{\partial \mathrm{x}}\right) + \hat{k}\left(\frac{\partial(-\mathrm{xz})}{\partial \mathrm{x}} - \frac{\partial(3\mathrm{y})}{\partial \mathrm{y}}\right)$$

$$\nabla \times \vec{F} = (z^2 + x)\hat{\imath} - (\mathrm{z} + 3)\hat{k}$$

3. Odnosi elementa površine paraboloida dS i elementa površine projekcije u xy ravnini dxdy

$$dS = \frac{|\nabla\Phi| dxdy}{|\nabla\Phi \cdot \hat{k}|} = \frac{dxdy}{|\hat{n} \cdot \hat{k}|} = \sqrt{x^2 + y^2 + 1} dxdy$$

Sada imamo sve potrebne elemente za računanje druge strane relacije Stokesovog teorema

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \cdot dS = \iint_S (xz^2 + x^2 + z + 3) dxdy =$$

$$= \iint_S \left[x \left(\frac{x^2 + y^2}{2} \right)^2 + x^2 + \frac{x^2 + y^2}{2} + 3 \right] dxdy =$$

$$= \int_{\varphi=0}^{2\pi} \int_{\rho=0}^{2} \left[\rho \cos\varphi \left(\frac{\rho^2}{2} \right)^2 + \rho^2 \cos^2\varphi + \frac{\rho^2}{2} + 3 \right] \rho \, d\rho \, d\varphi = 20\pi$$

I5-0910: 3. Rad sile

Odredite rad koji obavimo pomičući česticu jednom oko elipse C u xy ravnini (centar elipse nalazi se u ishodištu, velika poluos iznosi 4m, a manja 3m), ako djelujemo silom

$$\vec{F} = (3x - 4y + 2z) Nm^{-1} \hat{\imath} + (4x + 2y - 3z) Nm^{-1} \hat{\jmath} + (2xz - 4y^2 + z^2) Nm^{-2} \hat{k}$$

$$\vec{F} = (3x - 4y + 2z)\hat{\imath} + (4x + 2y - 3z)\hat{\jmath} + (2xz - 4y^2 + z^2)\hat{k}$$

$$\vec{F}_{xy\ ravnina} = (3x - 4y)\hat{\imath} + (4x + 2y)\hat{\jmath} + (2xz - 4y^2)\hat{k}$$

$$d\vec{r}_{xy\ ravnina} = dx\hat{\imath} + \mathrm{d}y\hat{\jmath}$$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \left[(3x - 4y)\hat{\imath} + (4x + 2y)\hat{\jmath} + (2xz - 4y^2)\hat{k} \right] \cdot (dx\hat{\imath} + \mathrm{d}y\hat{\jmath}) =$$

$$= \oint_C [(3x - 4y)dx + (4x + 2y)dy]$$

Pošto integrirano po elipsi, integraciju je lakše vršiti u polarnom koordinatnom sustavu pa uvodimo zamjenu koordinata i granica

$$x = 4\cos\varphi$$
$$y = 3\sin\varphi$$

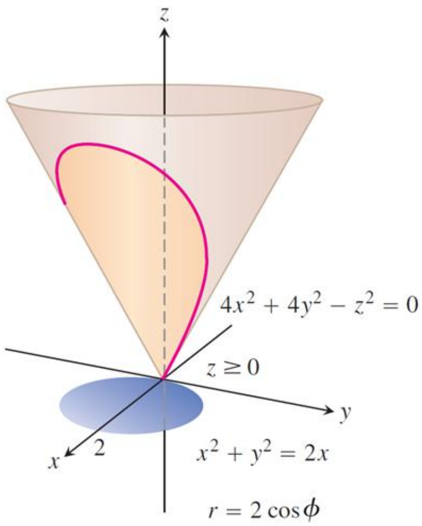
$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{\varphi=0}^{2\pi} \{[3(4\cos\varphi) - 4(3\sin\varphi)](-4\sin\varphi)d\varphi + [4(4\cos\varphi) + 2(3\sin\varphi)](3\cos\varphi)d\varphi\} =$$

$$= \oint_{\varphi=0}^{2\pi} (48 - 30\sin\varphi\cos\varphi)d\varphi =$$

$$= (48\varphi - 15\sin^2\varphi)\Big|_0^{2\pi} = 96\pi$$

I3-0910: 2. Moment tromosti

Odredite moment inercije oko z-osi tanke ljuske, konstantne plošne gustoće σ, isječene iz stošca $4x^2 + 4y^2 - z^2 = 0$, $z \geq 0$ cilindrom $x^2 + y^2 = 2x$ kao na slici dolje.



Moment inercije (tromosti) površine (ljuske) s obzirom na z-os iznosi:

$$I_z = \iint_S r^2 dm = \iint_S (x^2 + y^2)\sigma dS$$

Ako je S_{xy} jednoznačna projekcija glatke plohe S $[z = \varphi(x,y)$ na ravninu xOy (svaki pravac paralelan s osi Oz siječe plohu S samo u jednoj točki), tada vrijedi:

$$\iint_S f(x,y,z)dS = \iint_{S_{XY}} f(x,y,\varphi(x,y))\sqrt{1 + \varphi'^2_x(x,y) + \varphi'^2_y(x,y)}\, dxdy$$

Ako projiciramo isječeni dio stošca S na xy ravninu, dobit ćemo krug S_{xy} koji je jednoznačna projekcija od S. Općenito dvostruki integral u krivocrtnim koordinatama glasi:

$$\iint_{S_{xy}} f(x,y)dxdy = \iint_{S_{uv}} f[x(u,v),y(u,v)] \cdot |J| \cdot dudv$$

Dobiveni krug zapravo je dio koji isiječe dani cilindar iz xOy ravnine.

$x^2 + y^2 = 2x$
$x^2 - 2x + 1 + y^2 = 1$
$(x-1)^2 + y^2 = 1$

Krug ima središe u točki (1,0), a radijus mu je 1

Jednadžba ruba kruga u polarnom koordinatnom sustavu ($x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $J = \rho$) glasi: $\rho = 2\cos\varphi$

Iz jednadžbe stošca izvući ćemo eksplicitnu jednadžbu plohe S $[z = \varphi(x,y)]$:

$$\left.\begin{matrix} 4x^2 + 4y^2 - z^2 = 0 \\ z \geq 0 \end{matrix}\right\} \Rightarrow z = \sqrt{4x^2 + 4y^2} = \varphi(x,y)$$

$$I_z = \iint_S (x^2 + y^2)\sigma dS = \sigma \iint_{S_{XY}} (x^2 + y^2)\sqrt{1 + \varphi'^2_x(x,y) + \varphi'^2_y(x,y)}\, dxdy$$

$$I_z = \sigma \iint_{S_{XY}} (x^2 + y^2)\sqrt{1 + \left(\frac{4x}{\sqrt{4x^2 + 4y^2}}\right)^2 + \left(\frac{4y}{\sqrt{4x^2 + 4y^2}}\right)^2}\, dxdy = \sigma \iint_{S_{XY}} (x^2 + y^2)\sqrt{\frac{4x^2 + 4y^2 + 16x^2 + 16y^2}{4x^2 + 4y^2}}\, dxdy$$

$$I_z = \sigma \iint_{S_{XY}} (x^2 + y^2)\sqrt{\frac{20(x^2 + y^2)}{4(x^2 + y^2)}}\, dxdy = \sigma\sqrt{5} \iint_{S_{XY}} (x^2 + y^2)\, dxdy = \sigma\sqrt{5} \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2\cos\varphi} \rho^2 \cdot \rho \cdot d\rho$$

$$I_z = \sigma\sqrt{5} \int_{-\pi/2}^{\pi/2} d\varphi \left(\frac{\rho^4}{4}\Big|_0^{2\cos\varphi}\right) = \frac{\sigma\sqrt{5}}{4} \int_{-\pi/2}^{\pi/2} d\varphi(16\cos^4\varphi - 0) = 4\sqrt{5}\sigma \int_{-\pi/2}^{\pi/2} \cos^4\varphi\, d\varphi = \cdots$$

$$I_{cos} = \int \cos^4\varphi\, d\varphi = \int \cos^3\varphi \cdot \cos\varphi\, d\varphi = \left\{\begin{matrix} u = \cos^3\varphi & dv = \cos\varphi\, d\varphi \\ du = 3\cos^2\varphi \sin\varphi\, d\varphi & v = \sin\varphi \end{matrix}\right\} =$$

$$I_{cos} = \cos^3\varphi \cdot \sin\varphi + 3\int \sin^2\varphi \cdot \cos^2\varphi\, d\varphi = \cos^3\varphi \cdot \sin\varphi + 3\int (1 - \cos^2\varphi) \cdot \cos^2\varphi\, d\varphi$$

$$I_{cos} = \cos^3\varphi \cdot \sin\varphi + 3\int \cos^2\varphi\, d\varphi - 3I_{cos}$$

$$I_{cos} = \frac{1}{4}\left[\cos^3\varphi \cdot \sin\varphi + 3\int \frac{1 + \cos 2\varphi}{2} d\varphi\right]$$

$$\int_{-\pi/2}^{\pi/2} \cos^4\varphi\, d\varphi = \frac{1}{4}\left[\cos^3\varphi \cdot \sin\varphi + \frac{3}{4}(2\varphi + \sin 2\varphi)\right]\Bigg|_{-\pi/2}^{\pi/2} = \frac{1}{4}\left[0 + \frac{3}{4}(\pi + \pi)\right] = \frac{3}{8}\pi \Rightarrow \boldsymbol{I_z = \frac{3\sqrt{5}\pi\sigma}{2}}$$